

On Mixing Constructions with Algebraic Spacers

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1 Introduction.

Following [1] we recall a proof of the mixing for almost all Ornstein's stochastic rank one constructions (section 2), then we replace stochastic spacers by special algebraic ones [2] and in this new situation we deduce the mixing from the weakly mixing property (section 3).

Rank one construction is determined by h_1 , a sequence r_j of cuttings and a sequence \bar{s}_j of spacers

$$\bar{s}_j = (s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j)).$$

We recall its definition. Let our T on the step j is defined on a collection of disjoint sets (intervals)

$$E_j, TE_jT^2, E_j, \dots, T^{h_j-1}E_j$$

(T is not defined on the latest interval $T^{h_j}E_j$). We cut E_j into r_j sets (subintervals) of the same measure

$$E_j = E_j^1 \sqcup E_j^2 \sqcup E_j^3 \sqcup \dots \sqcup E_j^{r_j},$$

then for all $i = 1, 2, \dots, r_j$ we consider columns

$$E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j}E_j^i.$$

Adding $s_j(i)$ spacers we obtain a collection of disjoint intervals

$$E_j^i, TE_j^iT^2E_j^i, \dots, T^{h_j}E_j^i, T^{h_j+1}E_j^i, T^{h_j+2}E_j^i, \dots, T^{h_j+s_j(i)}E_j^i.$$

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Setting for all $i < r_j$

$$TT^{h_j+s_j(i)}E_j^i = E_j^{i+1}$$

we get $(j+1)$ -tower

$$E_{j+1}, TE_{j+1}T^2E_{j+1}, \dots, T^{h_{j+1}}E_{j+1},$$

where

$$E_{j+1} = E_j^1,$$

$$h_{j+1} + 1 = (h_j + 1)r_j + \sum_{i=1}^{r_j} s_j(i).$$

Step by step we define a construction T on a union X of all above intervals, assuming $\mu(X) = 1$.

On notations. We denote weak operator approximations by \approx_w , and \approx for strong ones. Θ is the orthogonal projection into the space of constant functions in $L_2(X, \mu)$. The expression $T^m \approx_w \Theta$ (for large m) means that T is mixing.

2 Stochastic constructions

D. Ornstein has proved [1] the mixing for almost all special rank one constructions. His approach can be presented in the following manner. Let $H_j \rightarrow \infty$, $H_j \ll r_j$. For uniformly distributed stochastic variables $a_j(i) \in \{0, 1, \dots, H_j\}$ we set

$$s_j(i) = H_j + a_j(i) - a_j(i+1).$$

Then for $m \in [h_j, h_{j+1})$

$$T^m = \hat{D}_1 T^m + \hat{D}_2 T^m + \hat{D}_3 T^m \approx_w \hat{D}_1 T^{k_1} P_1 + \hat{D}_2 T^{k_2} P_2 + \hat{D}_3 T^{k_3} P_3,$$

where

\hat{D}_i are operators of multiplication by indicators of certain sets $D_1, D_2, D_3 = X \setminus (D_1 \sqcup D_2)$, all \hat{D}_i (and k_i, P_i) depend on m ;

$k_1 = m - h_{j+1} - H_{j+1}$, $|k_2|, |k_3| < h_j$;

the operators P_i **for almost all constructions** T satisfy

$$P_1 \approx \sum_{n \in [-H_{j+1}, H_{j+1}]} c_{j+1}(n) T^n, \quad c_{j+1}(n) = \frac{H_{j+1} + 1 - |n|}{(H_{j+1} + 1)^2},$$

$$P_{2,3} \approx \sum_{n \in [-H_j, H_j]} c_j(n) T^n, \quad c_j(n) = \frac{H_j + 1 - |n|}{(H_j + 1)^2},$$

for all large m . As for the operators P_i , they satisfy

$$P_i^* P_i - T P_i^* P_i \approx_w 0,$$

this implies $P_i^* P_i \approx_w \Theta$ as T is ergodic (all rank one transformations are ergodic!), hence, $P_i \approx \Theta$. From $\|\hat{D}_i T^{k_i}\| \leq 1$ and $P_i \approx \Theta$ we get for a. a. constructions T

$$T^m \approx_w \Theta$$

for all large m , so T is mixing.

More details. Denote

$$S_j(k, N) = -NH_j + \sum_{i=k}^N s_j(i) = a_j(k) - a_j(k + N).$$

For example, as $m = Nh_j$ ($N \ll r_j$) we get

$$\begin{aligned} T^{-Nh_j} &\approx_w \frac{1}{r_j - N - 1} \sum_{k=1}^{r_j - N - 1} T^{\sum_{i=k}^N s_j(i)} = \\ &= \frac{1}{r_j - N - 1} \sum_{k=1}^{r_j - N - 1} T^{NH_j + S_j(k, N)} = \frac{1}{r_j - N - 1} \sum_{k=1}^{r_j - N - 1} T^{NH_j + a_j(k) - a_j(k + N)}. \end{aligned}$$

If

$$m = N(h_j + H_j) + v, \quad N \ll r_j, \quad NH_j \ll h_j, \quad v \ll h_j,$$

then

$$T^m \approx_w \frac{1}{r_j - N - 1} T^{k_2} \sum_{k=1}^{r_j - N - 1} T^{-a_j(k) + a_j(k + 2)}, \quad (1)$$

where $k_2 = v - NH_j$.

Generally we define D_1, D_2, D_3 :

$$D_1 = \bigsqcup_{i=0}^m T^i E_{j+1},$$

$$D_3 = (X \setminus D_1) \cap \bigsqcup_{i=0}^{k_2} T^i E_j,$$

$$h_j > k_2 = m - N(h_j + H_j),$$

$$D_2 = X \setminus (D_1 \cup D_2),$$

and use the following approximation:

$$T^m = \hat{D}_1 T^m + \hat{D}_2 T^m + \hat{D}_3 T^m \approx_w \hat{D}_1 T^{k_1} Q_1 + \hat{D}_2 T^{k_2} Q_2 + \hat{D}_3 T^{k_3} Q_3,$$

where $k_1 = m - h_{j+1} - H_{j+1}$, $k_3 + k_2 = h_j + H_j$, and

$$Q_1 = \frac{1}{r_{j+1} - 1} \sum_{i=1}^{r_{j+1}-1} T^{-a_{j+1}(i)+a_{j+1}(i+1)},$$

$$Q_2 = \frac{1}{r_j - N} \sum_{i=0}^{r_j-N} T^{-a_j(i)+a_j(i+N)},$$

$$Q_3 = \frac{1}{r_j - N - 1} \sum_{i=0}^{r_j-N-1} T^{-a_j(i)+a_j(i+N+1)}$$

(for the biggest N satisfied $Nh_j < m$).

Let's remark that sometimes (as in (1)) some D_i becomes of a small measure ($\|\hat{D}_i\| \approx 0$). This time the corresponding operator Q_i could be out of consideration.

If $Q_i \approx \Theta$ ($i = 1, 2, 3$), then T is mixing.

For almost all stochastic T for a vector $\{a_j(i)\}$, $i \in [1, r_j]$, the frequency

$$\frac{|\{i \in [1, r_j - N] : a_j(i) - a_j(i + N) = n\}|}{r_j - N}$$

is close to $c_j(n)$ (we recall that $H_j \ll r_j$). Here we assume that $N < (1 - \delta_j)r_j$ for $\delta_j \rightarrow 0$ very slowly. This explains the above approximations $Q_i \approx P_i$.

3 Algebraic spacers instead of stochastic ones

Now we present certain effective spacer sequences and another arguments to get

$$Q_i \approx \Theta.$$

Let r_j be prime, $r_j \rightarrow \infty$. We fix generators q_j in the multiplicative groups (associated with the sets $\{1, 2, \dots, r_j - 1\}$) of the fields \mathbf{Z}_{r_j} . For some sequence $\{H_j\}$, $H_j \geq r_j$, we define a spacer sequence

$$s_j(i) = H_j + \{q_j^i\} - \{q_j^{i+1}\}, \quad i = 1, 2, \dots, r_j - 1,$$

where $\{q^i\}$ denotes the residue modulo r_j .

Let $H_j = r_j$. To prove the mixing we apply two properties of the spacers: for $n < r_j$ we have

- (1) $-r_j \leq S_j(i, n) := \sum_{k=1}^n s_j(i+k) - nH_j \leq r_j$, $i = 1, 2, \dots, r_j - n - 1$;
- (2) for $i \in \{1, 2, \dots, r_j - n - 1\}$ all values $S_j(i, n)$ are different.

Since

$$\{q_j^i\} - \{q_j^{i+n}\} = S_j(i, n) = S_j(m, n) = \{q_j^m\} - \{q_j^{m+n}\}$$

implies

$$q^i - q^{i+n} = q^m - q^{m+n}, \quad q^i = q^m, \quad i = m, \quad (\text{mod } r_j)$$

we get the injectivity property (2).

LEMMA. *Let $r(j) > \delta r_j$ for a fixed real $\delta \in (0, 1)$, and $r(j) + n = r_j$. If T is weakly mixing, then (1), (2) imply*

$$Q(j) = \frac{1}{r(j)} \sum_{i=1}^{r(j)} T^{S_j(i, n)} \approx \Theta.$$

COROLLARY. *If a weakly mixing construction satisfies (1), (2), then it is mixing.*

Proof. We have to show that for any $f \in L_2^0$ one has $\|Q(j)f\| \rightarrow 0$. Otherwise there is a sequence j_k such that

$$Q(j_k) \rightarrow_w Q \neq \Theta. \quad (*)$$

Defining a measure η by the formula

$$\eta(A \times B) = \langle Q\chi_A, Q\chi_B \rangle_{L_2}$$

we see that $\eta = (T \otimes T)\eta$ and $\eta \ll \mu \times \mu$. The latter follows from

$$\mu(A)\mu(B) \leftarrow \langle Q'(j)\chi_A, Q'(j)\chi_B \rangle \geq c \langle Q(j)\chi_A, Q(j)\chi_B \rangle \rightarrow c\eta(A \times B),$$

where

$$Q'(j) = \frac{1}{2r_j} \sum_{i=-r_j}^{r_j} T^i, \quad c \geq \delta^2/4$$

(we remark that $|\{S_j(i, n) : 1 \leq i \leq r(j)\}| = r(j) \geq \delta r_j$, and $-r_j \leq S_j(i, n) \leq r_j$). Since T is weakly mixing, $T \otimes T$ is ergodic, so $\eta = \mu \times \mu$, $Q = \Theta$. This contradicts $(*)$ and shows that $\|Q(j)f\| \rightarrow 0$.

Replacing stochastic spacers by algebraic ones and providing the weakly mixing property (i. e. the absence of eigenfunctions) we get mixing constructions.

Are algebraic constructions weakly mixing? Note that the density of $\{S_j(i, 1) : 1 \leq i \leq r_j\}$ in $[-r_j, r_j]$ is approx 0.5. Taking this into account we can see that only the eigenvalue -1 could appear. There are several simple "ergodic" ways to conserve (1),(2) and eliminate an eigenfunction by adding a little spacer. We must avoid a situation in which for most i one has $(-1)^{S(i,1)} = (-1)^{h_j}$, hence, we should be out of the same parity for most of $\{q^i\} - \{q^{i+1}\}$. With a pleasure and our thanks to Oleg German we present his remark on "a parity of the parities" for certain pure constructions that provides them the weakly mixing property.

O.N. German's arguments. In 1962 Burgess proved that the minimal primitive root modulo a prime r is $O(r^{0.25+\varepsilon})$, where ε is a fixed arbitrarily small positive real number. Hence for each r_j large enough we may choose q_j to be less than $\sqrt{r_j}$. So, let r be a large prime number (particularly, r is supposed to be odd), and let q be a primitive root modulo r , such that $1 < q < \sqrt{r}$. Let us split the interval $[0, r)$ into the union of q intervals of length r/q and denote these intervals as follows

$$I_k = [kr/q, (k+1)r/q, k = 0, 1, \dots, q-1.$$

Let us also denote by σ_i the parity of the difference $\{q^i\} - \{q^{i+1}\}$, i.e.

$$\sigma_i = \{q^i\} - \{q^{i+1}\} \bmod 2,$$

and divide the set $M = \{1, 2, \dots, r-1\}$ into the the two subsets

$$M_0 = \{i \in M : \sigma_i = 0\}, \quad M_1 = \{i \in M : \sigma_i = 1\}.$$

Suppose q is odd. Then the parities of $\{q^i\}$ and $\{q^{i+1}\}$ coincide (i.e. $\sigma_i = 0$) if

and only if $\{q^i\} \in I_k$ with even k . In this case we have

$$||M_0| - |M_1|| \leq r/q + q - 1 < r/2 + \sqrt{r}.$$

Here we have made use of the fact that the numbers of integer points contained in two intervals of equal length differ at most by 1. Suppose q is even. Then $\sigma_i = 0$ if and only if $\{q^i\} \in I_k$ and the parity of $\{q^i\}$ coincides with that of k . Hence in this case we have

$$||M_0| - |M_1|| \leq q - 1 < \sqrt{r},$$

for in each I_k the numbers of even points and odd points differ at most by 1. Thus, if r is large, then in case of an odd q , both M_0 and M_1 contain almost $1/4$ of all the residues, and in case of an even q those portions make almost $1/2$.

References

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